Prof. Amador Martin-Pizarro
Übungen: Xier Ren

Albert-Ludwigs-Universität Freiburg
Sommersemester 2024

Topology

Problem Sheet 6
Deadline: 4 June 2024, 15h

Exercise 1 (3 Points)**.**
Show that the following conditions are equivalent for a topological space $(X, \mathcal{T})$:

a) The space is Hausdorff.

b) Every filter converges to at most one point.

c) Every ultrafilter converges to at most one point.

Exercise 2 (6 Points)**.**

a) Is the subset $[0, 1]$ compact in the Sorgenfrey line?

b) If A is a subset of the Sorgenfrey line, show the subset of A of non-limit points is countable.

c) Conclude that every compact subset of the Sorgenfrey line must be countable and hence with empty interior.

Hint: If A is uncountable, show that there is some n in $\mathbb{N}$ with $A \cap (n, \infty)$ uncountable. Deduce that A contains a strictly increasing sequence.

Exercise 3 (3 Points)**.**
Show that every compact Hausdorff space is *normal*: every two disjoint closed subset A and B can be separated by open subsets, that is, there are open subsets U and V with $A \subset U, B \subset V$ and $U \cap V = \emptyset$.

Exercise 4 (8 Points)**.**
Let $f : I \to X$ be a net in the topological space $(X, \mathcal{T})$, where I is a fixed directed partially ordered set. Set $A = \{x_\alpha\}_{\alpha \in I}$ with $a_\alpha = f(\alpha)$. Given an ultrafilter $\mathcal{U}$ on I, we say that the point x in X is a $\mathcal{U}$-limit of the net $(x_\alpha)_{\alpha \in I}$ if $\{\alpha \in I \mid x_\alpha \in U\}$ belongs to $\mathcal{U}$ for every neighbourhood U of x.

a) Show that $\mathrm{cl}(A)$ is the collection of all the $\mathcal{U}$-limits of $(x_\alpha)_{\alpha \in I}$, as $\mathcal{U}$ runs among all possible ultrafilters on I.

b) Assume now that $I = \mathbb{Z}$. For which ultrafilters does the net $n \mapsto (-1)^n$ has a limit in the euclidean line $\mathbb{R}$?

Assume now that $X = \mathbb{R}$ with the euclidean topology. A net $(x_\alpha)_{\alpha \in I}$ is $\mathcal{U}$-bounded if there is some $M \geq 0$ in $\mathbb{R}$ such that $\{\alpha \in I \mid |x_\alpha| \leq M\} \in \mathcal{U}$.

c) If the net $(x_\alpha)_{\alpha \in I}$ is $\mathcal{U}$-bounded by M, show that the set $Z = \{y \in \mathbb{R} \mid \{\alpha \in I \mid x_\alpha \leq y\} \in \mathcal{U}\}$ is not empty and bounded from below.

d) Conclude that the $\mathcal{U}$-bounded net $(x_\alpha)_{\alpha \in I}$ has a $\mathcal{U}$-limit x in the interval $[-M, M]$.

Die Übungsblätter können zu zweit eingereicht werden. Abgabe der Übungsblätter im entsprechenden Fach im Keller des mathematischen Instituts.